

Heat and Mass transfer

Heat Transfer from a Tapered Fin

Tapered Fin

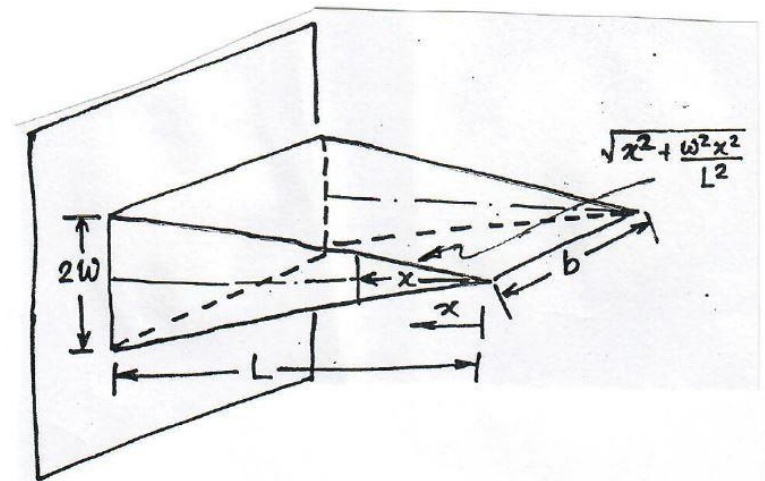
- Let us assume that the fin has the shape of the wedge.
- It is convenient to choose the origin of the coordinate system at the tip of the fin.

$$\text{➤ } S(x) = 2 b x \sqrt{1 + \frac{w^2}{L^2}} + \frac{2w}{L} x^2 ;$$

$$\text{➤ } \frac{dS/dx}{A} = \frac{2 b \sqrt{1 + \frac{w^2}{L^2}}}{2 w b x/L} + \frac{4 w x/L}{2 w b x/L} ;$$

$$\text{or, } \frac{S'}{A} = \frac{1}{x} \cdot \frac{L}{w} \cdot \sqrt{1 + \frac{w^2}{L^2}} + \frac{2}{b}$$

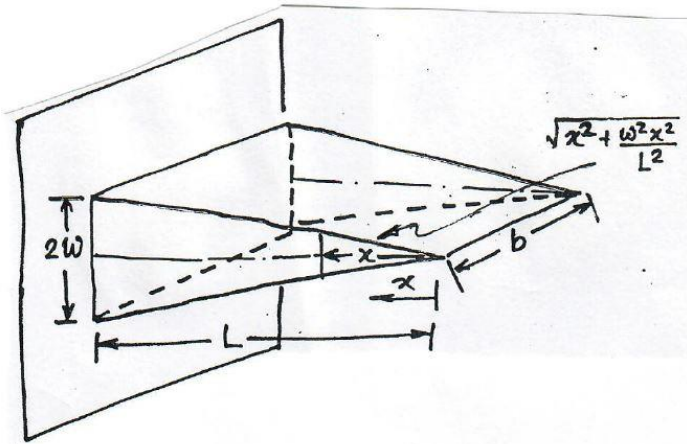
$$\text{or, } \frac{S'}{A} \approx \frac{1}{x} \cdot \frac{L}{w} + \frac{2}{b} \quad \text{if } \frac{w}{L} < 1$$



$$\text{➤ } A(x) = \frac{2w}{L} x b ;$$

$$\text{➤ } \frac{dA/dx}{A} = \frac{1}{x} ;$$

Tapered Fin



$$\frac{d^2 T}{dx^2} + \frac{dA/dx}{A} \frac{dT}{dx} - \frac{h}{kA} \frac{dS}{dx} (T - T_\infty) = 0$$

- The governing equation reduces to

$$\frac{d^2 T}{dx^2} + \frac{1}{x} \frac{dT}{dx} - \frac{h}{k} \left\{ \frac{1}{x} \cdot \frac{L}{w} + \frac{2}{b} \right\} (T - T_\infty) = 0$$

- The second term in the parenthesis is negligible compared to the first term and hence the equation reduces to:

$$\frac{d^2 T}{dx^2} + \frac{1}{x} \frac{dT}{dx} - \frac{h}{k} \cdot \frac{L}{w} (T - T_\infty) \cdot \frac{1}{x} = 0 \quad (\text{Justify the assumption})$$

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➤ Put $\theta = T - T_{\infty}$

$$\frac{d^2\theta}{dx^2} + \frac{1}{x} \frac{d\theta}{dx} - \frac{1}{x} \cdot \frac{h}{k} \cdot \frac{L}{w} \cdot \theta = 0$$

➤ Let $\rho = \beta x^{\gamma}$: $\frac{d\rho}{dx} = \beta \gamma x^{\gamma-1}$ or, $\frac{d\theta}{dx} = \frac{d\theta}{d\rho} \cdot \frac{d\rho}{dx} = \beta \gamma x^{\gamma-1} \frac{d\theta}{d\rho}$

➤ $\frac{d^2\theta}{dx^2} = \frac{d}{dx} \left[\frac{d\theta}{dx} \right] = \frac{d}{dx} \left[\beta \gamma x^{\gamma-1} \frac{d\theta}{d\rho} \right] = \beta \gamma (\gamma - 1) x^{\gamma-2} \frac{d\theta}{d\rho} + \frac{d^2\theta}{d\rho^2} \beta^2 \gamma^2 x^{2\gamma-2}$

So, $\frac{d^2\theta}{d\rho^2} \beta^2 \gamma^2 x^{2\gamma-2} + \frac{d\theta}{d\rho} \beta \gamma (\gamma - 1) x^{\gamma-2} + \beta \gamma x^{\gamma-2} \frac{d\theta}{d\rho} - \frac{1}{x} \cdot \frac{h}{k} \cdot \frac{L}{w} \theta = 0$

Or, $\beta^2 \gamma^2 x^{2\gamma-2} \frac{d^2\theta}{d\rho^2} + \beta \gamma^2 x^{\gamma-2} \frac{d\theta}{d\rho} - \frac{1}{x} \cdot \frac{hL}{kw} \theta = 0$

Or, $\beta^2 x^{2\gamma} \frac{d^2\theta}{d\rho^2} + \beta x^{\gamma} \frac{d\theta}{d\rho} - \frac{x}{\gamma^2} \cdot \frac{hL}{kw} \theta = 0$

Tapered Fin

➤ Now,

$$\rho^2 = \beta^2 x^{2\gamma} = \frac{x}{\gamma^2} \frac{hL}{kw}$$

$$\text{If } \gamma = \frac{1}{2}$$

$$\beta^2 = \frac{4hL}{kw} \Rightarrow \beta = 2 \sqrt{\frac{hL}{kw}}$$

$$\beta^2 x^{2\gamma} \frac{d^2 \theta}{d\rho^2} + \beta x^\gamma \frac{d\theta}{d\rho} - \frac{x}{\gamma^2} \cdot \frac{hL}{kw} \theta = 0$$

➤ Allows us to write:

$$\rho^2 \frac{d^2 \theta}{d\rho^2} + \rho \frac{d\theta}{d\rho} - \rho^2 \theta = 0$$

$$\text{Or, } \frac{d^2 \theta}{d\rho^2} + \frac{1}{\rho} \frac{d\theta}{d\rho} - \theta = 0$$

This equation is modified Bessel's equation and the solutions are called modified Bessel's functions.

Bessel's functions

➤ Bessel's equation of order n: $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$

➤ Solutions are:

- Bessel's function of first kind:

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m}$$

- Bessel's function of second kind:

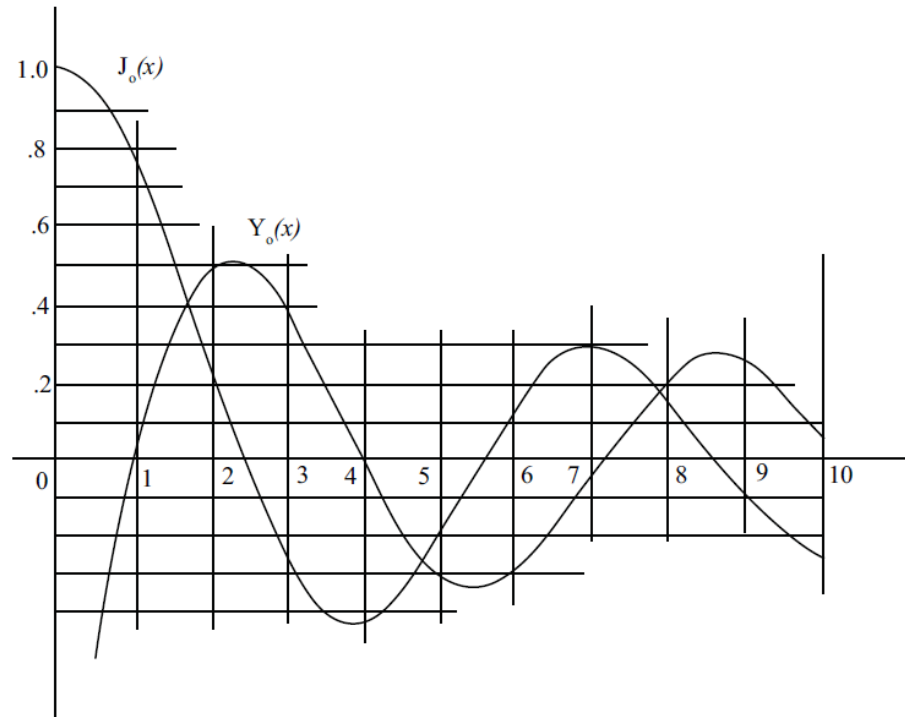
$$Y_n(x) = \frac{2}{\pi} \left\{ \ln \left(\frac{x}{2} \right) + \gamma \right\} J_n(x) - \frac{1}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{x}{2} \right)^{2m-n}$$

where $\gamma = 0.5772156$ is an Euler's constant

Bessel's functions

The complete solution of Bessel's equation of order zero is

$$y = AJ_0(x) + BY_0(x)$$



Bessel's functions of order zero

Modified Bessel functions

- Modified Bessel's equation of order n:
(Writing ix in place of x),

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + n^2)y = 0$$

- *Modified Bessel functions of the first kind* of order n and argument x , denoted by $I_n(x)$, and *modified Bessel functions of the second kind* of order n and argument x , denoted by $K_n(x)$, are defined, respectively, by the infinite series

$$I_n(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{2m+n}}{m! \Gamma(m+n+1)}$$

and

$$K_n(x) = \frac{\pi}{2 \sin n\pi} [I_{-n}(x) - I_n(x)] \quad (n \neq 0, 1, 2, \dots)$$

Modified Bessel functions

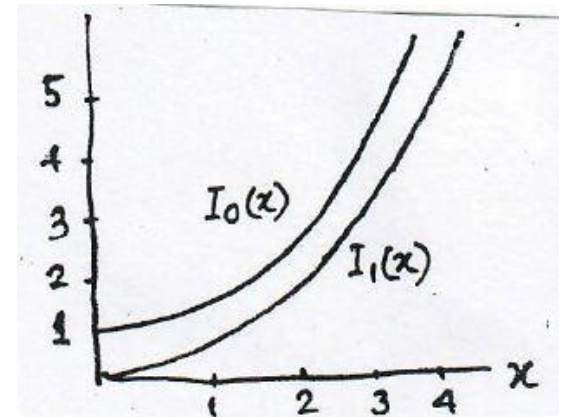
Or

$$K_n(x) = \lim_{m \rightarrow n} \frac{\pi}{2 \sin n\pi} [I_{-m}(x) - I_m(x)] \quad (n = 0, 1, 2, \dots)$$

$I_n(x)$ and $K_n(x)$ are real and positive when $n > -1$ and $x > 0$.

➤ A selected few of the numerous recurrence relationships involving the modified Bessel functions are

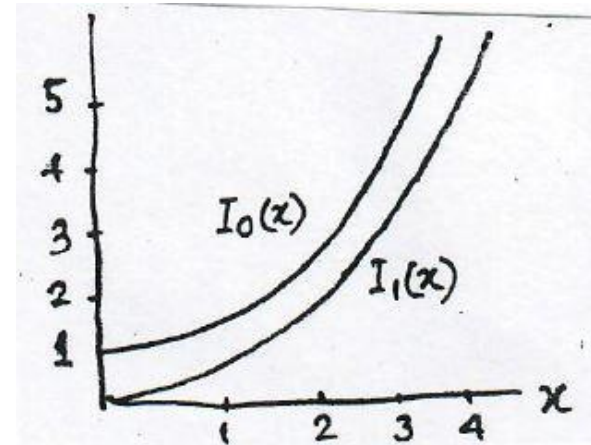
- $I_n(x) = (i)^{-n} J_n(ix)$
- $I_{-n}(x) = (i)^{-n} J_{-n}(ix)$
- $\frac{dI_n(x)}{dx} = I_{n-1}(x) - \frac{n}{x} I_n(x) = \frac{n}{x} I_n(x) + I_{n+1}(x)$



Modified Bessel functions

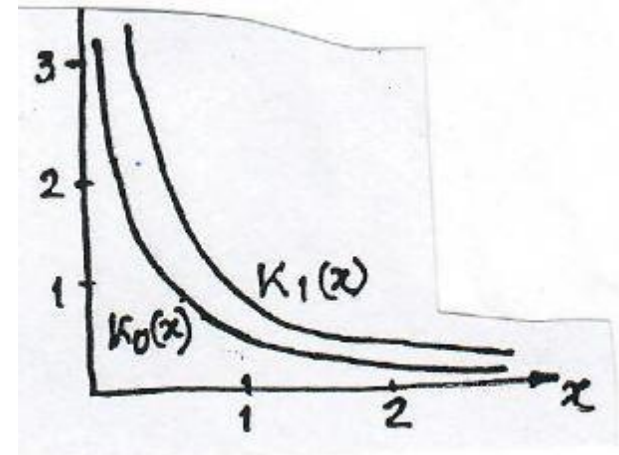
- $\frac{d}{dx} [x^n I_n(x)] = x^n I_{n-1}(x)$

- $\frac{d}{dx} [x^{-n} I_n(x)] = x^{-n} I_{n+1}(x)$



- $K_{-n}(x) = K_n(x) \quad (n = 0, 1, 2, 3, \dots)$

- $\frac{dK_n(x)}{dx} = \frac{n}{x} K_n(x) - K_{n+1}(x) = -K_{n-1}(x) - \frac{n}{x} K_n(x)$



Tapered Fin

- If $n = 0$; we get modified Bessel's equation of order 0

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - x^2 y = 0$$

$$\text{or, } \frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - y = 0$$

- Solutions are:

$$I_0(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{2m}}{m! \Gamma(m+1)} \quad \text{and} \quad K_0(x)$$

- Our Equation is

$$\frac{d^2 \theta}{d\rho^2} + \frac{1}{\rho} \frac{d\theta}{d\rho} - \theta = 0$$

- Solution of our equation: $\theta = C_1 I_0(\rho) + C_2 K_0(\rho)$ or, $\theta = C_1 I_0(\beta\sqrt{x}) + C_2 K_0(\beta\sqrt{x})$

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- At $x = 0$, $I_0 = 1$, $K_0 \rightarrow \infty \Rightarrow C_2 = 0$ since the temperature everywhere is finite

$$\theta = C_1 I_0(\beta\sqrt{x})$$

- At $x = L$, $\theta = \theta_0 \Rightarrow \theta_0 = C_1 I_0(\beta\sqrt{L})$ so, $C_1 = \theta_0 / I_0(\beta\sqrt{L})$

Therefore,

$$\theta = \theta_0 \frac{I_0(\beta\sqrt{x})}{I_0(\beta\sqrt{L})}$$

- We also find

$$\frac{d\theta}{dx} = \frac{\beta x^{-1/2}}{2} \theta_0 \frac{I_1(\beta\sqrt{x})}{I_0(\beta\sqrt{L})}$$

Tapered Fin

➤ The heat transfer rate is given by

$$q = +kA \left. \frac{d\theta}{dx} \right|_{x=L} = k 2wb \cdot \frac{\beta L^{-1/2}}{2} \frac{I_1(\beta\sqrt{L})}{I_0(\beta\sqrt{L})}$$

$$\text{or,} \quad q = k 2wb 2 \sqrt{\frac{hL}{kw}} \cdot \frac{L^{-1/2}}{2} \theta_0 \frac{I_1(\beta\sqrt{L})}{I_0(\beta\sqrt{L})}$$

$$\text{or,} \quad q = 2\theta_0 b \sqrt{hkw} \frac{I_1(\beta\sqrt{L})}{I_0(\beta\sqrt{L})}; \quad \text{where,} \quad \beta\sqrt{L} = 2L \sqrt{\frac{h}{kw}}$$

u	I ₀	u	I ₀	u	I ₀
(1)	(2)	(1)	(2)	(1)	(2)
0	1	2.0	2.280	4.0	11.30
0.1	1.003	2.1	2.446	4.1	12.32
0.2	1.010	2.2	2.629	4.2	13.44
0.3	1.023	2.3	2.830	4.3	14.67
0.4	1.040	2.4	3.049	4.4	16.01
0.5	1.063	2.5	3.290	4.5	17.48
0.6	1.092	2.6	3.553	4.6	19.09
0.7	1.126	2.7	3.842	4.7	20.86
0.8	1.167	2.8	4.157	4.8	22.79
0.8	1.213	2.9	4.503	4.9	24.91
1	1.266	3.0	4.881	5.0	27.24
1.1	1.326	3.1	5.294	5.1	29.79
1.2	1.394	3.2	5.747	5.2	32.58
1.3	1.469	3.3	6.243	5.3	35.65
1.4	1.553	3.4	6.785	5.4	39.01
1.5	1.647	3.5	7.378	5.5	42.69
1.6	1.750	3.6	8.028	5.6	46.74
1.7	1.864	3.7	8.739	5.7	51.17
1.8	1.990	3.8	9.517	5.8	56.04
1.9	2.128	3.9	10.37	5.9	61.38

Notes : Reference: SPIEGEL (1974); For low values of u, I₀ can be approximated by :

$$I_0(u) = 1 + \frac{u^2}{2^2} + \frac{u^4}{2^2 * 4^2}; \text{ For large values of u, } I_0 \text{ can be approximated by : } I_0(u) = \exp(u)/\sqrt{2*\pi*u}.$$

Thank you